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ON ACCELERATION WAVES IN ANISOTROPIC THERMOELASTIC MEDIA TAKING ACCOUNT OF FINITENESS OF THE HEAT PROPAGATION VELOCITY

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The propagation of acceleration waves in an anisotropic thermoelastic medium is studied. It is shown that taking account of the finiteness of the heat distribution velocity results in the appearance of four kinds of accelaration waves, whose velocities and damping coefficients depend in an essential way on the direction of wave surface propagation. A comparison between the velocities and damping coefficients of plane acceleration waves in a zinc crystal, obtained with and without the finiteness of the heat propagation velocity taken into account, is presented.

The papers [1, 2] are devoted to the influence of the coupling of the strain and temperature fields on the nature of wave propagation in a homogeneous isotropic body in the case of an infinite heat distribution velocity. A number of features due to coupling of the fields is obtained therein, and it is shown in particular that weak and strong discontinuities damp out, and the order of damping is determined by an exponential factor.

Taking account of finiteness of the heat distribution velocity results in the appearance of two kinds of longitudinal waves whose propagation velocities depend in an essential manner on the velocity of the heat perturbation [3, 4].

1. Let us write down the system of equations governing the dynamical behavior of a thermoelastic anisotropic medium in which the heat is propagated at a finite velocity

$$q_{j,j} + c_\varepsilon \theta^{\cdot} + T_0 \beta_{ij} \varepsilon_{ij}^{\cdot} = 0 \qquad (1.1)$$

$$\tau q_j^{\cdot} + q_j = -K_{ij}\theta_{,i} \qquad (1.2)$$

$$\sigma_{ij,j} = \rho u_i^{\cdot\cdot} \qquad (1.3)$$

$$\varepsilon_{ij} = {}^1/_2 (u_{i,j} + u_{j,i}) \qquad (1.4)$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta \qquad (1.5)$$

Here q_j are the heat flux vector components, $\theta = T - T_0$ is the body temperature, T_0 is the body temperature in the natural state, c_ε is the specific heat for constant strain,

$\beta_{ij} = C_{ijkl}\alpha_{kl}$, α_{kl} are the linear thermal expansion coefficients, C_{ijkl} are the isothermal stiffness coefficients of the anisotropic material, ε_{ij} is the strain tensor, K_{ij} are the heat conduction coefficients, τ is the relaxation time of the heat flux, σ_{ij} is the stress tensor, u_i the displacement components, ρ the density; the Latin subscripts take on the values 1, 2, 3, the dots over the quantities denote the time derivative, and the subscript after the comma denotes the derivative with respect to the appropriate coordinate.

The energy conservation equation (1.1), the law of Fourier taking account of the heat flux inertia (1.2), the motion equation (1.3), the Cauchy formula (1.4), and the generalized Duhamel-Neumann relationship (1.5) are a system of nineteen equations in the unknowns q_i, θ, σ_{ij}, u_i, ε_{ij}.

Henceforth, the acceleration wave will be understood to be the isolated surface on which the stresses, the velocities, the temperature, and the heat flux are continuous, but some of their partial derivatives are discontinuous.

Taking into account that the quantities q_j are continuous on the wave surface, we obtain from Eqs. (1.1) – (1.3) and the relationships (1.4), (1.5) differentiated with respect to time (the square brackets denote the difference between values of corresponding quantities on different sides of the discontinuity surface)

$$[q_{j,j}] + c_\varepsilon [\theta^{\cdot}] + T_0\beta_{ij} [\varepsilon_{ij}^{\cdot}] = 0, \quad \tau [q_j^{\cdot}] = - K_{ij} [\theta_{,i}] \qquad (1.6)$$
$$[\sigma_{ij,j}] = \rho [v_i^{\cdot}], \quad 2 [\varepsilon_{ij}^{\cdot}] = [v_{j,i}] + [v_{j,i}], \ [\sigma_{ij}] = C_{ijkl} [v_{k,l}] - \beta_{ij} [\theta]$$

where $[v_i]$ is the jump in the displacement velocity.

Using tne first-order kinematic and geometric compatibility conditions on the discontinuity surface, we find from (1.6)

$$\rho c^2\lambda_i = S_{ik}\lambda_k + b_i c\mu, \qquad (c^2 - a^2)\,\mu = T_0 c_\varepsilon^{-1} c b_k \lambda_k \qquad (1.7)$$
$$S_{ik} = C_{ijkl}\nu_j\nu_l, \quad b_i = \beta_{ij}\nu_j, \quad a^2 = K_{mn}\nu_m\nu_n (\tau c_\varepsilon)^{-1}$$

Here ν_i is the unit vector normal to the surface, c is the propagation velocity of this surface, a^2 is the square of the velocity of the thermal perturbation for the uncoupled problem, λ_i, μ are quantities characterizing the jumps in the first derivatives of the displacement and temperature rates, respectively

$$[v_i^{\cdot}] = - \lambda_i c, \quad [v_{i,j}] = \lambda_i\nu_j, \quad [\theta^{\cdot}] = - \mu c, \quad [\theta_{,i}] = \mu\nu_i$$

The existence condition for nontrivial solutions of the system (1.7) which is homogeneous relative to λ_i and μ, determines four velocities in the general case, and therefore, four kinds of acceleration waves in the anisotropic medium. Upon compliance with this condition, only three equations in the system (1.7) remain linearly independent. Selecting μ as the free unknown, we obtain

$$\lambda_i = \frac{d_i}{d}\mu c, \qquad d = | S_{ij} - \delta_{ij}\rho c^2 | \qquad (1.8)$$

Here δ_{ij} is the Kronecker delta, d_i are determinants obtained from d by replacing the i-th column with a column from $(- b_i)$. In the case of an infinite velocity of heat distribution in the body $(\tau = 0)$, the relationships (1.7) and the condition that the determinant of the system (1.7) vanish, are rewritten as

$$\rho\, c^2 \lambda_i = S_{ik}\lambda_k, \ d = 0$$

For specified ν_i these equations can be considered as the equations governing the real principal values ρc_1^2, ρc_2^2, ρc_3^2 and the orthogonal principal directions $l_i^{(1)}$, $l_i^{(2)}$, $l_i^{(3)}$ of the symmetric tensor of the second rank $C_{ijkl}\nu_j\nu_l$. Then the quantities λ_i can be represented as

$$\lambda_i = \lambda l_i, \quad \lambda = \sqrt{\lambda_i\lambda_i} \tag{1.9}$$

on each of the three wave surfaces. For simplicity, the superscript indicating the ordinal number of the wave surface is omitted on the quantities λ_i, λ, l_i.

2. Let us investigate the change in the characteristic values of the acceleration waves in the propagation process.

Differentiating (1.1) and (1.3) with respect to t, (1.2) with respect to x_j and (1.5) with respect to t and x_j and taking their difference on both sides of the wave surface, we obtain

$$[\dot q_{j,j}] + c_\varepsilon[\theta^{\cdot\cdot}] + T_0\beta_{ij}[\dot v_{i,j}] = 0, \quad \tau[\dot q_{j,j}] + [q_{j,j}] + K_{ij}[\theta_{,ij}] = 0 \tag{2.1}$$

$$[\dot\sigma_{i,j}] = \rho[v_i^{\cdot\cdot}], \quad [\dot\sigma_{ij,j}] = C_{ijkl}[v_{k,lj}] - \beta_{ij}[\dot\theta_{,j}] \tag{2.2}$$

Eliminating the quantity $[\dot q_{j,j}]$ from (2.1), and $[\delta_{ij,j}]$ from (2.2), we find

$$-[q_{j,j}] + \tau c_\varepsilon[\theta^{\cdot\cdot}] + T_0\tau\beta_{ij}[\dot v_{i,j}] - K_{ij}[\theta_{,ij}] = 0 \tag{2.3}$$

$$\rho[v_i^{\cdot\cdot}] = C_{ijkl}[v_{k,lj}] - \beta_{ij}[\theta_{,j}^{\cdot}] \tag{2.4}$$

Using the second-order kinematic and geometric compatibility conditions [5, 6], we obtain a differential equation to determine the characteristic quantities from (2.3) and (2.4)

$$\rho c\,\frac{\delta\lambda_i^2}{\delta t} + \lambda_i^2\frac{\delta c}{\delta t} + C_{ijkl}\lambda_i g^{\alpha\beta}\lambda_{k,\alpha}(\nu_l x_{j,\beta} + \nu_j x_{l,\beta}) - \tag{2.5}$$
$$C_{ijkl}\lambda_i\lambda_k g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{j,\beta}x_{l,\tau} - b_i\lambda_i\frac{\delta\mu}{\delta t} + \beta_{ij}\lambda_i g^{\alpha\beta}(c\mu)_{,\alpha}x_{j,\beta} +$$
$$(T_0\tau)^{-1}K_{ij}g^{\alpha\beta}\mu\mu_{,\alpha}(\nu_i x_{j,\beta} + \nu_j x_{i,\beta}) - (T_0\tau)^{-1}\mu^2 K_{ij}g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{i,\beta}x_{j,\tau} -$$
$$\mu b_i\frac{\delta\lambda_i}{\delta t} + \mu\beta_{ij}g^{\alpha\beta}(c\lambda_i)_{,\alpha}x_{j,\beta} + T_0^{-1}c_\varepsilon c\,\frac{\delta\mu^2}{\delta t} +$$
$$T_0^{-1}c_\varepsilon\mu^2\frac{\delta c}{\delta t} + (T_0\tau)^{-1}a^2c_\varepsilon c^{-1}\mu^2 = 0$$

Here $g^{\alpha\beta}$ is the contravariant metric tensor of the wave surface, $b_{\alpha\sigma}$ are coefficients of the second quadratic form of this surface, y_α are curvilinear coefficients on the surface, the indices α, β, σ, τ $= 1.2$, the $\delta/\delta t$ denotes δ-differentiation with respect to t [5]. Expressing the quantity λ_i in terms of μ in (2.5) by means of (1.8), we obtain

$$A_1\frac{\delta\mu^2}{\delta t} + B_{1\alpha}\mu_{,\alpha}^2 + D_1\mu^2 = 0, \quad A_1 = \rho c^3 d_k d_k d^{-2} + T_0^{-1}c_\varepsilon c^{-1}a^2 \tag{2.6}$$
$$B_{1\alpha} = \frac{1}{2}c^2 C_{ijkl}d_i d_k d_k d^{-2}g^{\alpha\beta}(\nu_l x_{j,\beta} + \nu_j x_{l,\beta}) + c^2 d_i d^{-1}g^{\alpha\beta}x_{j,\beta}\beta_{ij} +$$
$$(T_0\tau)^{-1}K_{ij}\nu_i x_{j,\beta}g^{\alpha\beta}$$
$$D_1 = (T_0\tau)^{-1}c_\varepsilon c^{-1}a^2 - g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{l,\beta}x_{j,\tau}[(T_0\tau)^{-1}K_{lj} + C_{ijkl}c^2 d_i d_k d^{-2}] +$$
$$\rho c\delta(d_k d_k d^{-2}c^2)/\delta t + (d_k d_k d^{-2}c^2 + T_0^{-1}c_\varepsilon)\delta c/\delta t - b_i\delta(d_i d^{-1}c)/\delta t +$$
$$C_{ijkl}d_i d^{-1}c g^{\alpha\beta}(d_k d^{-1}c)_{,\alpha}(\nu_l x_{j,\beta} + \nu_j x_{l,\beta}) +$$
$$\beta_{ij}d_i d^{-1}c g^{\alpha\beta}c_{,\alpha}x_{j,\alpha} + g^{\alpha\beta}\beta_{ij}(c^2 d_i d^{-1})_{,\alpha}x_{j,\beta}$$

Expressing the quantities λ_i in (2.5) in terms of λ by means of (1.9) for an infinite velocity of heat propagation ($\tau = 0$), we obtain

$$A_2 \frac{\delta\lambda^2}{\delta t} + B_{2\alpha}\lambda^2_{,\alpha} + D_2\lambda^2 = 0, \quad A_2 = \rho c_1 B_{2\alpha} = C_{ijkl}l_i l_k \nu_l x_{j,\beta} g^{\alpha\beta} \qquad (2.7)$$

$$D_2 = \rho \frac{\delta c}{\delta t} + C_{ijkl}(l_i l_k)_{,\alpha}\nu_l x_{j\beta} g^{\alpha\beta} - C_{ijkl} l_i l_k x_{l\tau} x_{j\beta} g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} + cT_0 (b_k l_k)^2 (K_{ij}\nu_i\nu_j)^{-1}$$

It follows from (2.6) and (2.7) that a change in the characteristic values of the acceleration waves of all kinds depends essentially on the direction of wave surface propagation. In the general case, the solution of (2.6) and (2.7) is fraught with awkward calculations. Hence, let us henceforth limit ourselves to the consideration of plane waves. If a Cartesian orthogonal coordinate system is chosen on the wave surface, then (2.6) and (2.7) are written as (no summation is made over γ)

$$\frac{\delta X_\gamma}{\delta t} + \Omega_{\gamma\alpha} X_{\gamma,\alpha} + W_\gamma X_\gamma = 0, \quad \gamma = 1, 2 \quad (X_1 = \mu^2, \; X_2 = \lambda^2) \qquad (2.8)$$

Here

$$\Omega_{1\alpha} = [{}^1/{}_2 c^2 C_{ijkl} d_i d_k d^{-2} (\nu_l \cos(j, \alpha) + \nu_j \cos(l, \alpha) + c^2 d_i d^{-1} \beta_{ij} \cos(j, \alpha) + (T_0\tau)^{-1} K_{ij}\nu_i \cos(j, \alpha)] (\rho c^3 d_k d_k d^{-2} + T_0^{-1} c_\varepsilon c^{-1} a^2)^{-1}$$

$$W_1 = (T_0\tau)^{-1} c_\varepsilon c^{-1} a^2 (\rho c^3 d_k d_k d^{-2} + T_0^{-1} c_\varepsilon c^{-1} a^2)^{-1}$$

$$\Omega_{2\alpha} = C_{ijkl} l_i l_k \nu_l \cos(j, \alpha) (\rho c)^{-1}, \quad W_2 = T_0 (b_k l_k)^2 (\rho K_{ij}\nu_i\nu_j)^{-1}$$

Let us integrate (2.8) along the characteristics. The characteristic surfaces have the form ($y_{\alpha 0}$ is an arbitrary constant)

$$y_\alpha = \Omega_{\gamma\alpha} t + y_{\alpha 0} \qquad (2.9)$$

The quantities X_γ vary along the characteristic according to the law ($X_{\gamma 0}$ is some constant)

$$X_\gamma = X_{\gamma 0} \exp\{-W_\gamma t\} \qquad (2.10)$$

It is seen from (2.9), (2.10) that the quantities $\Omega_{\gamma\alpha}$ characterize the deviation of the wave tubes from the normal vector to the wave surface, and W_γ determines the damping of the perturbation X_γ along the wave tubes.

3. As an illustration, let us examine plane wave damping in a hexagonal zinc crystal [7, 8].

For simplicity, we shall consider the normal vector ν_i orthogonal to the x_1-axis, which is a second-order axis of symmetry [7], i.e. $\nu_1 = 0$. We select an orthogonal Cartesian coordinate system in the plane of the wave such that the y_1-axis is parallel to the x_1-axis. The condition that the determinant of the system (1.7) vanish is in this case

$$\rho c^2 - S_{11} = 0, \; c^6 - mc^4 + nc^2 + p = 0 \qquad (3.1)$$

$$m = a^2 + \rho^{-1} [S_{22} + S_{33} + T_0 c_\varepsilon^{-1} (b_2^2 + b_3^2)]$$

$$n = a^2 \rho^{-1} (S_{22} + S_{33}) + \rho^{-2} [S_{22}S_{33} - S_{23}^2 + (T_0 c_\varepsilon^{-1}) \times (S_{33}b_2^2 + S_{22}b_3^2 - 2 S_{23}b_2 b_3)]$$

$$p = a^2 \rho^{-2} (S_{23}^2 - S_{22}S_{33})$$

It follows from (1.7) that a purely transverse wave ($\mu = 0, \; \lambda_i\nu_i = 0$) is propagated at the velocity $c_1 = S_{11}^{1/2}\rho^{1/2}$. The other kinds of waves, whose velocities are determined by the second equation in (3.1), will be classified as follows: the wave which is purely transverse along the principal directions $\nu_2 = 0$, $\nu_3 = 0$ will be called quasi-transverse (its velocity is denoted by c_2) and the other two, being longitudinal along these directions, will be called quasi-longitudinal [9] (their velocities are c_3, c_4).

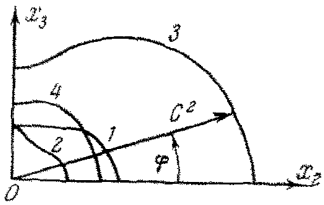

Fig. 1

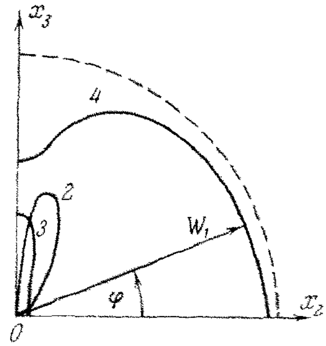

Fig. 2

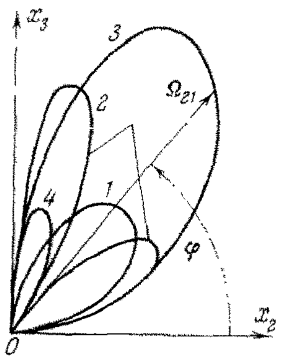

Fig. 3

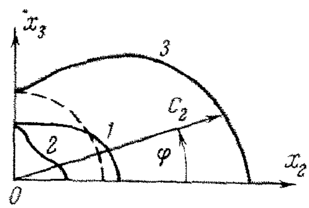

Fig. 4

The dependence of the squares of the velocities on the angle φ ($\nu_2 = \cos\varphi$, $\nu_3 = \sin\varphi$) is shown in Fig. 1 (here and henceforth only the first quadrant is shown because of symmetry). Curves *1–4* correspond to the velocities $C_1 - C_4$. The value of the relaxation time was taken to be $\tau = 0{,}5 \cdot 10^{-11}$ sec [10].

It is seen from Fig. 1 that the transverse wave velocities c_1 and c_2 coincide along the principal direction $\nu_2 = 0$. For $\varphi \approx 37°$ the velocities of the transverse c_1 and the quasi-longitudinal c_4 waves coincide. For the transverse wave being propagated with velocity c_1, the sole quantity λ_1 different from zero is determined by the equation

$$\rho c \frac{\delta\lambda_1}{\delta t} + \lambda_{1,2}(C_{1212} - C_{1313})\nu_2\nu_3 = 0 \qquad (3.2)$$

Therefore, the transverse wave does not damp out, and the wave tubes deviate from the normal vector. In the case of multiple velocities ($c_1 = c_2$) $\lambda_3 = \mu = 0$ and λ_1 and λ_2 satisfy the equation

$$2\rho c \frac{\delta\lambda_\gamma}{\delta t} + C_{\gamma j\beta l}\lambda_{\beta,\alpha}[\nu_l \cos(j, \alpha) + \nu_j \cos(l, \alpha)] = 0 \qquad (3.3)$$

It can be seen that the second member in (3.3) is zero, hence, the solution (3.3) yields $\lambda_1 = \text{const}$, $\lambda_2 = \text{const}$. In the case of multiple velocities ($c_1 = c_4$) the quantity λ_1 satisfies (3.2), and λ_2, λ_3 and μ are determined from (2.8).

Presented in Figs. 2 and 3 are dependences of the damping cooefficients W_1 and the quantity Ω_{12} characterizing the deviation of the wave tubes from the normal vector, on the angle φ. It is seen that the quasi-longitudinal wave being propagated with velocity c_4 damps out most rapidly, while the quasi-transverse wave has minimal damping (curve *2* in Fig. 2 is magnified twentyfold). It should be noted that the wave tubes only deviate in the x_2x_3 plane, and the quantity $\Omega_{\gamma 1}$ equals

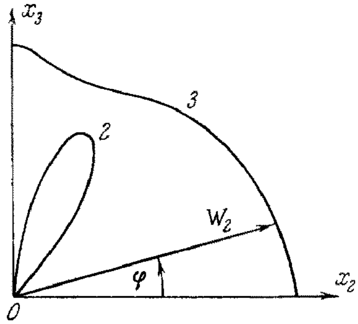

Fig. 5

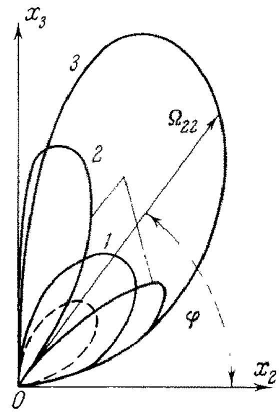

Fig. 6

zero. The maximal deviation is for the quasi-longitudinal wave with velocity c_3. The deviation of the quasi-transverse wave changes sign starting with $\varphi \approx 52°$ and becomes negative. The quantity Ω_{12} changes sign for all kinds of waves upon passing through $\varphi \approx \pi/2$.

In the case of an infinite velocity of heat distribution in the zinc crystal, the transverse wave with velocity c_1 coincident with the transverse wave velocity in the $\tau \neq 0$ case is propagated, as is also the quasi-transverse with velocity c_2 and the quasi-longitudinal with velocity c_3.

The dependence of the velocities, the damping coefficients W_2 and the quantity Ω_{22} on the angle φ are presented in Figs. 4 – 6, respectively. Curves *1—3* correspond to the velocities c_1 — c_3. Curve *2* in Fig. 5 is magnified sixfold.

It is seen from the graphs that the finiteness of the heat propagation velocity essentially influences the nature of plane wave propagation, especially the behavior of their damping coefficients.

In conclusion, let us consider the uncoupled problem taking account of the finiteness of the heat propagation velocity. In this case, four kinds of acceleration waves exist in the crystal. Three of them are propagated with the velocities of elastic waves (Fig. 4), and one with the velocity $c = a$. For elastic waves the change in the characteristic quantity λ_3 is determined by (2. 8), in which it is necessary to set $T_0 = 0$. It is seen that these waves do not damp out, but the deviation of the wave tubes agrees exactly with the $\tau = 0$ case (Fig. 6). In the $c = a$ case, the change in the characteristic quantity μ is determined by the equation

$$2\tau c_\varepsilon a \frac{\delta\mu}{\delta t} + K_{ij} g^{\alpha\beta} \mu_{,\alpha} (\nu_i x_{j,\beta} + \nu_j x_{i,\beta}) + a c_\varepsilon \mu = 0$$

The dependence of the square of the velocity a, the damping coefficient W, and the deviation of the wave tubes Ω on the angle φ are shown by dashes in Figs. 4, 2, 6, respectively. It is seen that the thermal wave velocity and damping in a zinc crystal depend weakly on the propagation direction.

Thus, a characteristic singularity of acceleration wave propagation in an anisotropic medium is the deviation of the wave tubes from the normal vector. For $\tau = 0$ a second quasi-longitudinal wave appears which damps out more rapidly than the first. The relaxation time τ turns out to exert substantial influence on the nature of quasi-longitudinal and quasi-transverse wave propagation.

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ON DYNAMIC EFFECTS IN AN ELASTIC HALF-SPACE UNDER "THERMAL IMPACT"

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The general uncoupled dynamical problem of thermoelasticity for a half-space under the condition of a thermal impact with a finite rate of change in temperature on its boundary is solved by the method of principal (fundamental) functions within the framework of a generalized theory of heat conduction.

An elastic steel half-space is analyzed as an illustration. The problem on thermal stresses originating in an elastic half-space due to thermal impact produced by a jump change in temperature on the boundary was first analyzed in [1]. Since the temperature change on the boundary occurs at a finite rate, it is gene-